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Counting the Total Induced Matchings for Recursive Trees

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Abstract. An induced matching of the graph G is a matching which forms an induced subgraph of 1-regular in G. Induced matching is widely used in computer networks, such as communication network testing, concurrent transmission of messages, secure communication channels etc. Counting problem on the total number of induced matchings was introduced by Liu et al.(2021). In view of its computational difficulty, the total number of induced matchings for restricted graph is therefore of interest. By combination method and characteristic equation we get the closed formulas of the total number of induced matchings for some recursive trees.

Keywords: Induced matching, recursive tree, the total number of induced matchings

1. Introduction

This paper only considers undirected simple graph. Let G be a graph with the vertex set $V(G)$ and edge set $E(G)$. For any $v \in V(G)$, let $N_G(v) = \{u \in V(G) | uv \in E(G)\}$ and $N_G[v] = N_G(v) \cup \{v\}$. The vertex $v \in V(G)$ is called a leaf if $d_G(v) = 1$. The edge incident with a leaf is known as a pendant edge. Say that $M(\subseteq E(G))$ is a matching of G if any two edges of M have no common vertices in G. A matching M of G is called induced if the subgraph of G induced by the $V[M]$ is 1-regular, i.e. distinct edges in M have distance at least 2 in G. Stockmeyer et al.(1982) first considered the induced matching and proved computational hardness of the maximum induced matching problem [1]. From [1-3] we know that induced matching has certain applications in communication network, wireless ad hoc network and broadcast network. Thus, many scholars are committed to the research of induced matching problem. In particular, there have been many important achievements in the study on induced matching number, for example, Romeo, Hibi, Panda et al. ([4-6]) calculated the induced matching number of some special graphs; Ajayi et al.([7]) determined the some bounds on the maximum induced matching numbers of certain grids; Hirano et al. ([8]) studied the relationship between matching number, induced matching number and dimension of edge ideals of graph. Recently, counting problem on the total number of induced matchings was introduced by Liu et al. (2021) ([9]). In view of its computational difficulty, the total number of induced matchings for restricted graph is therefore of interest. Since recursive tree is widely used in network science, especially in the problem of finding effective algorithms. This paper mainly concern the total number of induced matchings for the recursive trees which constructed from paths and stars.

The following Definitions and Lemmas will be used in sequeal.

Definition 1.1[9] Let $im(G,k)$ denote the number of induced matching of graph G containing k edges. Set $im(G,0)=1$. The total of induced matchings of graph G is defined as

$$iz(G) = \sum_{k \geq 0} im(G,k) .$$

Definition 1.2[10] For $n \geq 2$, let $f_n = f_{n-1} + f_{n-3}$ such that $f_0 = 1, f_1 = 1$. Then the number f_n is called the n-th Fibonacci-Narayana number.

Lemma 1.1[9] Let $G_{1,} G_{2,} \cdots G_r$ be all components of G. Then $iz(G) = iz(G_1) iz(G_2) \cdots iz(G_r)$.

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Lemma 1.2[9] Let v be a vertex of the graph G.

(1) If $d_G(v)=1$ and $uv\in E(G)$, then $iz(G)=iz(G-v)+iz(G-N_G[u])$.

(2) If $d_G(v)=2$ and $uv,vw\in E(G)$, then

$$iz(G)=iz(G-v)+iz(G-N_G(u)-N_G(v))+iz(G-N_G(v)-N_G(w)).$$

Let P_n be the path of order n. S_m denotes the star with m vertices. Clearly, $iz(P_n)=f_n$ and $iz(S_m)=m$. $V^1_{n,m}$ denotes the graph obtained from P_n and S_m by attaching the vertex of order m-1 in S_m at each vertex of P_n. We use $V^2_{n,m}(m\geq 2)$ to denote the graph obtained from P_n and S_m by attaching a leaf of S_m at each vertex of P_n. Obviously, $V^1_{n,m}=V^2_{n,m}$ if $m=2$. We first give the recurrence relations on the total induced matching number for $V^1_{n,m}$ and $V^2_{n,m}(m\geq 2)$. Then by using homogeneous characteristic equation we get the exact calculation formulas of the total number of induced matchings for the recursive trees $V^1_{n,m}$ and $V^2_{n,m}(m\geq 2)$.

2. Main Results and Proofs

Theorem 2.1 For the recursive tree $V^1_{n,m}$, the total of induced matchings of $V^1_{n,m}$ is as follow:

$$iz(V^1_{n,m})=c_1(p+q+\frac{1}{3})^n+c_2(p\omega+q\omega^2+\frac{1}{3})^n+c_3(p\omega^2+q\omega+\frac{1}{3})^n,$$

where $p=\sqrt[3]{-\frac{1}{2}b+\sqrt{\frac{1}{4}b^2+\frac{1}{27}a^3}}$, $q=\sqrt[3]{-\frac{1}{2}b-\sqrt{\frac{1}{4}b^2+\frac{1}{27}a^3}}$, $a=\frac{2}{3}-m$, $b=-\frac{20}{27}-\frac{m}{3}$,

$$c_1=\frac{((p\omega^2+q\omega+\frac{1}{3})(p\omega+q\omega^2+\frac{1}{3})+m(p+q+\frac{4}{3}))}{(p\omega^2+q\omega-(p+q))(p\omega+q\omega^2-(p+q))},\ c_2=\frac{(-(p+q+\frac{1}{3})(p\omega^2+q\omega+\frac{1}{3})-m(\omega p+q\omega^2-\frac{2}{3}+2m))}{(p\omega+q\omega^2-(p+q))(p\omega^2+q\omega-(p\omega+q\omega^2))},$$

$$c_3=\frac{((p+q+\frac{1}{3})(p\omega+q\omega^2+\frac{1}{3})+m(p\omega^2+q\omega+\frac{4}{3}))}{(p\omega^2+q\omega-(p+q))(p\omega^2+q\omega-(p\omega+q\omega^2))}$$ and $\omega=(-1+\sqrt{3}i)/2$.

Proof. By Lemmas 1.1-1.2 we have

$$iz(V^1_{n,m})=iz(V^1_{n-1,m})+(m\text{-}1)iz(V^1_{n-2,m})+iz(V^1_{n-3,m}) \tag{1}$$

where $iz(V^1_{1,m})=iz(S_m)=m$ and $iz(V^1_{0,m})=1$.

Let $g_n=iz(V^1_{n,m})$. Then (1) implies that the following recursive relation:

$$\begin{cases} g_n=g_{n-1}+(m-1)g_{n-2}+g_{n-3}, \\ g_0=1,\ g_1=m,\ g_2=2m. \end{cases} \tag{2}$$

The corresponding characteristic equation of (2) is

$$x^3-x^2+(1-m)x-1=0. \tag{3}$$

Set $y=x-1/3$. From (3) we get

$$y^3+ay+b=0\text{ , where } a=\frac{2}{3}-m \text{ and } b=-\frac{20}{27}-\frac{m}{3}. \tag{4}$$

Suppose $y=z-a/3z$. Thus (4) implies that $z^6+bz^3\text{-}a^3/27=0$. Solving this equation we obtain that

$$z^3=-\frac{1}{2}b\pm\sqrt{\frac{1}{4}b^2+\frac{1}{27}a^3}. \tag{5}$$

Since $\frac{1}{4}b^2 - \frac{1}{27}a^3 > 0$. Then $p = \sqrt[3]{-\frac{1}{2}b + \sqrt{\frac{1}{4}b^2 + \frac{1}{27}a^3}}$ is a solution of (5). Thus the solutions of (5) are $p, p\omega$ and $p\omega^2$, where $\omega = (-1+\sqrt{3}i)/2$.

In addition, it is known from simple calculation that $-a/3p = \sqrt[3]{-\frac{1}{2}b - \sqrt{\frac{1}{4}b^2 + \frac{1}{27}a^3}}$.

Let $q = \sqrt[3]{-\frac{1}{2}b - \sqrt{\frac{1}{4}b^2 + \frac{1}{27}a^3}}$. According to $y = x - 1/3$ and $y = z - a/3z$, the solutions of (1) are

$$x_1 = p + q + \frac{1}{3},\ x_2 = p\omega + q\omega^2 + \frac{1}{3} \text{ and } x_3 = p\omega^2 + q\omega + \frac{1}{3}.$$

Therefore, $g_n = c_1x_1^{\,n} + c_2x_2^{\,n} + c_3x_3^n$ where c_1, c_2 and c_3 are the constant coefficients. Also by (2) we know that

$$\begin{cases} c_1 + c_2 + c_3 = 1, \\ c_1x_1 + c_2x_2 + c_3x_3 = m, \\ c_1x_1^2 + c_2x_2^2 + c_3x_3^2 = 2m. \end{cases}$$

Solving the above equations we obtain that $c_1 = \frac{(x_3x_2 - m(x_2 + x_3 - 2))}{(x_3 - x_1)(x_2 - x_1)}$, $c_2 = \frac{(-x_1x_3 + m(x_1 + x_3 - 2m))}{(x_2 - x_1)(x_3 - x_2)}$ and $c_3 = \frac{(x_1x_2 - m(x_1 + x_2 - 2))}{(x_3 - x_1)(x_3 - x_2)}$. The proof is complete.

Theorem 2.2 For the recursive tree $V_{n,m}^2$, the total of induced matchings of $V_{n,m}^2$ is

$$iz(V_{n,m}^2) = c_1(u + v + \frac{m-1}{3})^n + c_2(u\omega + v\omega^2 + \frac{m-1}{3})^n + c_3(u\omega^2 + v\omega + \frac{m-1}{3})^n,$$

where $u = \sqrt[3]{-\frac{1}{2}t + \sqrt{\frac{1}{4}t^2 + \frac{1}{27}s^3}}$, $v = \sqrt[3]{-\frac{1}{2}t - \sqrt{\frac{1}{4}t^2 + \frac{1}{27}s^3}}$, $s = \frac{2 - m - m^2}{3}$, $t = \frac{2(1-m)^3}{27} - \frac{m^2 + m - 2}{3}$,

$c_1 = \frac{(u\omega^2 + v\omega - (2m+1)/3)(u\omega + v\omega^2 - (2m+1)/3)}{(u\omega^2 + v\omega - u - v)(u\omega + v\omega^2 - u - v)}$, $c_2 = \frac{((2m+1)/3 - u - v)(u\omega^2 + v\omega - (2m+1)/3)}{(u\omega + v\omega^2 - u - v)(u\omega^2 + v\omega - u\omega - v\omega^2)}$,

$c_3 = \frac{((2m+1)/3 - u - v)((2m+1)/3 - u\omega - v\omega^2)}{(u\omega^2 + v\omega - u - v)(u\omega^2 + v\omega - u\omega - v\omega^2)}$ and $\omega = (-1+\sqrt{3}i)/2$.

Proof. By Lemma 1.1 and Lemma 1.2, we have

$$iz(V_{n,m}^2) = (m-1)[iz(V_{n-1,m}^2) + iz(V_{n-2,m}^2) + iz(V_{n-3,m}^2)]. \tag{6}$$

where $iz(V_{2,m}^2) = m^2$, $iz(V_{1,m}^2) = iz(S_m) = m$ and $iz(V_{0,m}^1) = 1$.

Let $w_n = iz(V_{n,m}^2)$. Then from (6) we get the following recursive relation

$$\begin{cases} w_n = (m-1)(w_{n-1} + w_{n-2} + w_{n-3}), \\ w_0 = 1,\ w_1 = m,\ w_2 = m^2. \end{cases} \tag{7}$$

So the corresponding characteristic equation of (7) is

$$x^3 - (m-1)(x^2 + x + 1) = 0. \tag{8}$$

Let $y = x + (1-m)/3$. Then by (8) we have

$$y^3 + sy + t = 0, \text{ where } s = \frac{2 - m - m^2}{3} \text{ and } t = \frac{2(1-m)^3}{27} - \frac{m^2 + m - 2}{3}. \tag{9}$$

Suppose $y = z - s/3z$. Then (9) implies that $z^6 + tz^3 - s^3/27 = 0$. Solving this equation we get

$$z^3 = -\frac{1}{2}t \pm \sqrt{\frac{1}{4}t^2 + \frac{1}{27}s^3}. \tag{10}$$

Since $\frac{1}{4}t^2 + \frac{1}{27}s^3 > 0$. Then $u = \sqrt[3]{-\frac{1}{2}t + \sqrt{\frac{1}{4}t^2 + \frac{1}{27}s^3}}$ is a solution of (10). Thus the solutions of (10) are $u, u\omega$ and $u\omega^2$, where $\omega = (-1 + \sqrt{3}i)/2$.

In addition, it is known from simple calculation that $-s/3u = \sqrt[3]{-\frac{1}{2}t - \sqrt{\frac{1}{4}t^2 + \frac{1}{27}s^3}}$.

Let $v = \sqrt[3]{-\frac{1}{2}t - \sqrt{\frac{1}{4}t^2 + \frac{1}{27}s^3}}$. According to $y = x + (1-m)/3$ and $y = z - s/3z$, the solutions of (6) are

$$x_1 = u + v + \frac{m-1}{3},\ x_2 = u\omega + v\omega^2 + \frac{m-1}{3} \text{ and } x_3 = u\omega^2 + v\omega + \frac{m-1}{3}.$$

Therefore, $w_n = c_1 x_1^n + c_2 x_2^n + c_3 x_3^n$ where c_1, c_2 and c_3 are the constant coefficients. Also by (7) we know that

$$\begin{cases} c_1 + c_2 + c_3 = 1, \\ c_1 x_1 + c_2 x_2 + c_3 x_3 = m, \\ c_1 x_1^2 + c_2 x_2^2 + c_3 x_3^2 = m^2. \end{cases}$$

Solving the above equations we obtain that $c_1 = \frac{(x_3 - m)(x_2 - m)}{(x_3 - x_1)(x_2 - x_1)}$, $c_2 = \frac{(m - x_1)(x_3 - m)}{(x_2 - x_1)(x_3 - x_2)}$ and $c_3 = \frac{(m - x_1)(m - x_2)}{(x_3 - x_1)(x_3 - x_2)}$. The proof is complete.

3. Conclusions

Using the recurrence relation of the induced matching number, the corresponding characteristic equation and Cardano method, we have studied the counting problem on the total induced matchings for two kinds recursive trees. The explicit representations of the total number of induced matchings for these recursive trees are obtained(Theorems 2.1-2.2). Obviously, these methods are suitable for solving the induced matching number of general recursive tree networks.

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